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Method of Interpolation Factorization (MIF) for the Solution of Two-Dimensional Diffraction Problems

Vladimir Volman and Jacques Gavan

***Abstract*—A new method for exact solution of two-dimensional diffraction problems is presented.**

I. Introduction

One of the effective numerical-analytical methods for the solution of a wide class of diffraction problems is the modified method of residues [1], [2]. The principal difficulties in this method appear when we begin the construction of a meromorphic function. Let's consider a new method for the solution of this problem. For clearness we shall consider TEM-analysis of a strip line (Fig. 1). As shown in [2], this problem leads to the construction of a meromorphic function $F(\omega)$, which satisfies the following conditions:

1) $F(\omega)$ has simple poles for $\omega = \alpha_{n1}$, where $n = 1, 2, \cdots, \infty$, and for $\omega = 0$;
2) $F(\alpha_{nj}) + \lambda_{nj} F(-\alpha_{nj}) = 0$, for $n = 1, 2, \cdots, \infty, j = 2, 3$; λ_{nj} and α_{nj} are known values [2].
3) $F(\omega)$ has the asymptotic behavior $|\omega \angle^{-3/2}$ for $|\omega| = \infty$;
4) the residue of $F(\omega)$ for $\omega = 0$ is equal to (-1),

II. Presentation of the Problem

Let's introduce a function similar to those described in [2]

$$F(\omega) = -\frac{\exp(\nu\omega)}{\omega} \prod_{n=1}^{\infty} \frac{(1-\omega/\alpha'_{n2})(1-\omega/\alpha'_{n3})}{(1-\omega/\alpha_{n1})},$$

$$\nu = [a_3 \ln(a_1/a_3) + a_2 \ln(a_1/a_2)]/\pi, \tag{1}$$

where α'_{n2} and α'_{n3} are unknown zeros of $F(\omega)$.

$F(\omega)$ satisfies in such form the 4th condition and fulfill the asymptotic behavior $|\omega|^{-3/2}$ for the following conditions [2]:

$$\alpha'_{nj} = n\pi/a_j, \qquad n \to \infty \text{ and } j = 2, 3, \tag{2}$$

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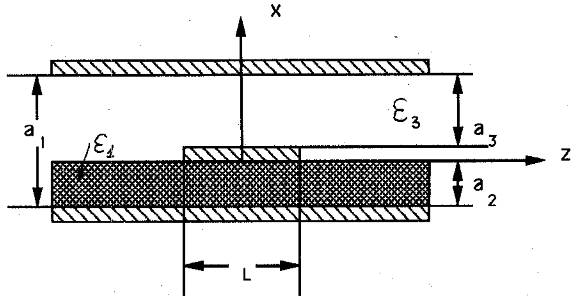

Fig. 1. View of strip line.

where a_j are known constants [2] and $a_1 = a_2 + a_3$. Therefore,

$$F(\omega) = -\frac{\exp(\nu\omega)}{\omega} R_{3M}(\omega) \prod_{n=M+1}^{\infty} \frac{(1-\omega a_2/n\pi)(1-\omega a_3/n\pi)}{(1-\omega a_1/n\pi)}, \tag{3}$$

where

$$R_{3M}(\omega) = \prod_{n=1}^{M} \frac{(1-\omega/\alpha'_{n2})(1-\omega/\alpha'_{n3})}{(1-\omega/\alpha_{n1})}, \tag{4}$$

and M is a number from which the asymptotic value presented in (2) may be used. Let us introduce the rational fractional function

$$G_0(\omega) = \frac{R_{3M}(\omega)}{R_{3M}(-\omega)}\bigg|_{\omega=\alpha_m} = \prod_{n=1}^{M}$$
$$\cdot \frac{(1+\omega/\alpha_{n1})(1-\omega/\alpha'_{n2})(1-\omega_3\alpha'_{n3})}{(1-\omega/\alpha_{n1})(1+\omega/\alpha'_{n2})(1+\omega/\alpha_{n3})}\bigg|_{\omega=\alpha_m}$$
$$= \{\lambda_m^{(0)}\}_1^{3M},$$
$$\{\alpha_m\}_1^{3M} = \{-\alpha_{m1}, \alpha_{m2}, \alpha_{m3},\}_1^M, \ \{\lambda_m^{(0)}\}_1^{3M}$$
$$= \{0, \lambda_{m2}, \lambda_{m3}\}_1^M \tag{5}$$

It is evident that

$$G_0(-\omega) = 1/G_0(\omega). \tag{6}$$

Let U_ω be the class of rational fractional functions satisfying (6). Then the considering problem leads to an interpolation problem in order to define the function $G_0(\omega)$ from the class U_ω, which is equal to

$$G_0(\alpha_m) = \{\lambda_m^{(0)}\}_1^{3M}. \tag{7}$$

III. Build Up of the Function $F(\omega)$

Consider the rational fractional function of class U_ω

$$G_0(\omega) = \frac{(\alpha_1 - \omega)G_1(\omega) + (\alpha_1 + \omega)G_0(\alpha_1)}{(\alpha_1 + \omega) + (\alpha_1 - \omega)G_0(\alpha_1)G_1(\omega)} \tag{8}$$

where $G_1(\omega) \in U_\omega$ and at the point $\omega = \alpha_1 G_0(\alpha_1)$ is equal to λ_1 independently of the selection of $G_1(\omega)$.

The values $G_1(\omega)$ are chosen at the points $\alpha_m (3M \geqslant m \geqslant 2)$ so that the (7) is satisfied not only at the point α_1 but at all the points $\alpha_m (m \geqslant 2)$. According to (8)

$$G_1(\alpha_m) = \frac{(\lambda_m^{(0)} - \lambda_1^{(0)})(\alpha_1 + \alpha_m)}{(1 - \lambda_m^{(0)}\lambda_1^{(0)})(\alpha_1 - \alpha_m)} = \{\lambda_m^{(1)}\}_2^M. \tag{9}$$

i.e., it's necessary to solve the same interpolation problem but for the function $G_1(\omega)$ of $m \geqslant 2$. If we continue this process up to the nth step

$$G_{n-1}(\omega) = \frac{(\alpha_{n-1} - \omega)G_n(\omega) + (\alpha_{n-1} + \omega)G_{n-1}}{(\alpha_{n-1} + \omega) + (\alpha_{n-1} - \omega)G_{n-1}G_n(\omega)}, \quad n \geqslant 2,$$

$$G_n(\alpha_m) = \frac{(\lambda_m^{(n-1)} - \lambda_n^{(n-1)})(\alpha_n + \alpha_m)}{(1 - \lambda_m^{(n-1)}\lambda_n^{(n-1)})(\alpha_n - \alpha_m)} = \{\lambda_m^{(n)}\}_{n+1}^M, \tag{10}$$

where $G_n(\omega) \in U_\omega$ and $G_{n-1} = G_{n-1}(\alpha_n)$. This process require $3M$th steps. If we substitute $G_1(\omega), G_2(\omega), \cdots, G_n(\omega)$ from (10) into (8) on the last step the following expression is obtained

$$G_0(\omega) = \frac{P_{3M}(\omega) + Q_{3M}(\omega)G_{3M+1}(\omega)}{Q_{3M}(-\omega) + P_{3M}(-\omega)G_{3M+1}(\omega)} \tag{11}$$

where $P_{3M}(\omega)$ and $Q_{3M}(\omega)$ are known polynomials of order $3M$ and $G_{3M+1}(\omega) \in U_\omega$.

According to (5), however the nominator and denominator of the function $G_0(\omega)$ must be polynomials of order $3M$ and $G_0(\omega) \in U_\omega$. It's not difficult to check these conditions and show that they are satisfied if in (11) we put $G_{3M+1}(\omega) = 1$. As $R_{3M}(0) = 1$ (see (4)) then

$$R_{3M}(\omega) = \frac{[P_{3M}(\omega) + Q_{3M}(\omega)]}{[P_{3M}(0) + Q_{3M}(0)]} \Big/ \prod_{m=1}^{M} (1 - \omega^2/\alpha_{m1}^2), \tag{12}$$

Let's present the $(n-1)$th and nth steps of equation (11)

$$G_0(\omega) = \frac{P_{n-2}(\omega) + Q_{n-2}(\omega)G_{n-1}(\omega)}{Q_{n-2}(-\omega) + P_{n-2}(-\omega)G_{n-1}(\omega)} = \frac{P_{n-1}(\omega) + Q_{n-}(\omega)G_n(\omega)}{Q_{n-1}(-\omega) + P_{n-1(-\omega)}G_n(\omega)}. \tag{13}$$

Introducing (10) into (13) and comparing the nominators of the given expressions we obtain the following recurrent equations

$$P_{n+1}(\omega) = (\alpha_n + \omega)[P_n(\omega) + G_n Q_n(\omega)],$$
$$Q_{n+1}(\omega) = (\alpha_n - \omega)[G_n P_n(\omega) + Q_n(\omega)],$$
$$P_0(\omega) = 0, \quad Q_0(\omega) = 1,$$
$$P_{n+1}(0) + Q_{n+1}(0) = \alpha_n(1 + G_n)[P_n(0) + Q_n(0)]. \tag{14}$$

Summing and substracting these equations we obtain

$$A_{n+1}(\omega) = A_n(\omega) + (\omega/\alpha_n)[(1 - G_n)/(1 + G_n)]B_n(\omega),$$
$$B_{n+1}(\omega) = [(1 - G_n)/(1 + G_n)]B_n(\omega) + (\omega/\alpha_n)A_n(\omega),$$
$$A_n(\omega) = 2\,[P_n(\omega) + Q_n(\omega)]/[P_n(0) + Q_n(0)],$$
$$B_n(\omega) = [P_n(\omega) - Q_n(\omega)]/[P_n(0) + Q_n(0)],$$
$$A_0(\omega) = 1, \quad B_0(\omega) = -1 \tag{15}$$

Since

$$A_{n+2}(\omega) = A_{n+1}(\omega) + (\omega/\alpha_{n+1})[(1 - G_{n+1})/(1 + G_{n+1})]B_{n+1}(\omega), \tag{16}$$

we can eliminate $B_n(\omega)$ and $B_{n+1}(\omega)$ from (15) and (16). After transformations we have the following relations

$$A_{n+2}(\omega) = c_{n+1}A_{n+1}(\omega) + d_{n+1}(\omega)A_n(\omega),$$
$$A_1(\omega) = 1 - (\omega/\alpha_1)(1 - G_0)/(1 + G_0),$$
$$c_{n+1} = 1 + (\alpha_n/\alpha_{n+1})(1 - G_{n+1})/(1 + G_{n+1}),$$
$$d_{n+1}(\omega) = (\omega^2 - \alpha_n^2)[(1 - G_{n+1})/(1 + G_{n+1}]/(\alpha_n/\alpha_{n+1}) \tag{17}$$

and according to (4), (12) and (15) we can write

$$R_{3M}(\omega) = \frac{A_{3M}(\omega)}{\prod_{m=1}^{M} (1 - \omega^2/\alpha_{mi}^2)} \tag{18}$$

If we put $y_{n+1}(\omega) = A_{n+1}(\omega)/A_n(\omega)$ then from (17) it is followed

$$y_{n+2}(\omega) = c_{n+1} + d_{n+1}(\omega)/y_{n+1}(\omega),$$
$$A_{n+2}(\omega) = \prod_{s=1}^{n+2} y_s(\omega) \quad \text{and}$$
$$y_{n+2} = c_{n+1} + \frac{d_{n+1}}{c_n +} \frac{d_n}{c_{n-1} +} \cdots \frac{d_1}{y_1} \tag{19}$$

which is a continued fractional function.

In all expressions we can reach the limit $M = \infty$ if the succession $\{\alpha_m\}_1^{3M}$ is arranged in an ordered form as (5). Then

$$A(\omega) = \lim_{M \to \infty} A_{3M}(\omega) = \prod_{s=1}^{\infty} y_s(\omega). \tag{20}$$

In the particular case of the strip line problem we can write

$$F(\omega) = -\frac{\exp(\nu\omega)}{\omega} \cdot \frac{A_{3M}(\omega)\prod_{m=1}^{M}(1 - \omega^2 a_1^2/m^2\pi^2)/(1 - \omega^2/\alpha_{m1}^2)}{\prod_{m=1}^{M}(1 + \omega a_1/m\pi)(1 - \omega a_2/m\pi)(1 - \omega a_3/m\pi)} \cdot \frac{\Gamma(1 - \omega a_1/\pi)}{\Gamma(1 - \omega a_2/\pi)\Gamma(1 - \omega a_3/\pi)} \tag{21}$$

where $\Gamma(z)$ is a Gamma-function. From (17) the following expressions are obtained

$$A_0(\omega) = 1, \quad A_1(\omega) = 1 - (\omega/\alpha_1)(1 - G_0)/(1 + G_0),$$
$$A_2(\omega) = c_1A_1(\omega) + d_1(\omega),$$
$$A_3(\omega) = (c_1c_3 + d_2(\omega))A_1(\omega) + c_2d_1(\omega),$$
$$A_4(\omega) = (c_1c_2c_3 + c_3d_2(\omega) + c_1d_3(\omega))A_1(\omega) + (c_2c_3 + d_3(\omega))d_1(\omega) \tag{22}$$

and so on.

Therefore analytical expression for $F(\omega)$ can be obtained at every step of the solution. It's obvious that the described algorithm is very simple and can be realized using only an elementary computer program.

IV. Calculation Results

The computation process can be simplified by transforming equation (21) and using known expressions for Gamma-function we have

$$\frac{\Gamma(1 - \omega a_1/\pi)}{\Gamma(1 - \omega a_2/\pi)\Gamma(1 - \omega a_3/\pi)} = \frac{\Gamma(1 + \omega a_1/\pi)}{\Gamma(1 + \omega a_2/\pi)\Gamma(1 + \omega a_3/\pi)} \cdot \frac{a_1 \sin(\omega a_2)\sin(\omega a_3)}{\omega \sin(\omega a_1)a_2a_3} \tag{23}$$

But

$$\frac{\Gamma(1 + \omega a_1/\pi)}{\Gamma(1 + \omega a_2/\pi)\Gamma(1 + \omega a_3/\pi)} = \prod_{m=1}^{\infty}\left(1 + \frac{\omega^2 a_2a_3}{m\pi(m\pi + \omega a_1)}\right), \tag{24}$$

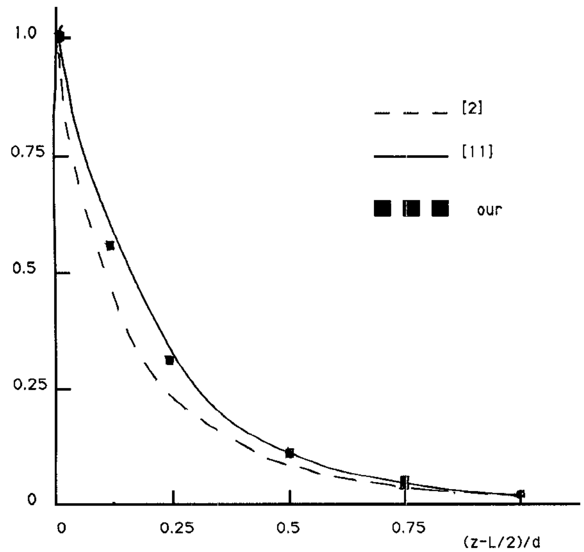

Fig. 2. The potential distribution at $x \to a_3$, $a_2 = a_1/\sqrt{1.56}$, $L/a = 1$, $\epsilon_1 = \epsilon_3 = 1$.

$$V_{3m}(\omega) = \prod_{m=1}^{M} (1 + \omega a_1/m\pi)(1 - \omega a_2/m\pi)(1 - \omega a_3/m\pi)$$
$$= \prod_{m=1}^{M} \left(1 - \frac{\omega^2(a_1a_2 + a_3^2)}{(m\pi)^2} + \frac{\omega^3 a_1a_2a_3}{(m\pi)^3}\right). \quad (25)$$

Due to the expression

$$F(\omega) = -\frac{a_1 \sin(\omega a_2) \sin(\omega a_3)}{\omega^2 \sin(\omega a_1) a_2 a_3} \frac{A_{3M}(\omega)}{V_{3M}(\omega)} \cdot \frac{\prod_{m=1}^{M} (1 - \omega^2 a_1^2/m^2\pi^2)/(1 - \omega^2/\alpha_{m1}^2)}{\prod_{m=1}^{\infty} \left(1 + \frac{\omega^2 a_2 a_3}{m\pi(m\pi + \omega a_1)}\right) e^{-\nu\omega}}. \quad (26)$$

In the simplest case $\alpha_{n1} = n\pi/a_1$ when $\epsilon_{r1} = \epsilon_{r2} = 1$ [2] and the residue of $F(\omega)$ for $\omega = \alpha_{n1}$ is equal

$$R_F(\alpha_{n1}) = (-1)^n \frac{\sin(n\pi a_2/a_1)}{(n\pi a_2/a_3)} \frac{\sin(n\pi a_3/a_1)}{(n\pi a_3/a_1)} \frac{A_{3M}(n\pi/a_1)}{V_{3M}(n\pi/a_1)} \Bigg/ \prod_{m=1}^{\infty} \left(1 + \frac{n^2 a_2 a_3/a_1}{m(m+n)}\right) e^{-\nu n\pi/a_1}. \quad (27)$$

Using (27) the potential distribution is calculated and compared to the results of [2] and exact solution [11]. The coincidence is excellent with exact solution as seen from Fig. 2. Note that all results are obtained when

$$\frac{A_{3M}(n\pi/a_1)}{V_{3M}(n\pi/a_1)} = \frac{1 - (na_2/a_1)\,\mathrm{cth}\,(\pi L/(2a_2))}{1 - (na_2/a_1)} \quad (28)$$

V. Conclusion

The described method is quite effective. It allows one to solve the problems practically in closed form and to realize all the required calculations using small computers only. The most interesting diffraction problems solved by this method are: a thick semi-infinite plate [3], infinite periodic corrugated structure [4], echelett [5] and other structures as presented in [6]–[10]. Analytical expressions for all mentioned cases were obtained.

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Optimal Microwave Source Distributions for Heating Off-Center Tumors in Spheres of High Water Content Tissue

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***Abstract*—A surface distribution of electric dipoles can be used to represent a multi-element microwave hyperthermia applicator for non-invasive heating of off-center targets within a spherical high-water-content tissue volume—such as the head. This paper presents a method for finding the optimal surface distributions for delivering maximum power for arbitrarily located deep tumors in such a uniform spherical volume. The resulting focused power dissipation pattern for any tumor location has a global maximum at the tumor, and also is the largest spherical volume for which no healthy tissue is overheated. The optimization uses spherical field harmonics, centered at the tumor target, summed with suitable complex weights to iteratively minimize surface power. Once the best field distributions are derived, the current sources which generate these distributions are determined. The resulting excitations represent the theoretically ideal spherical microwave hyperthermia configuration for which no physical applicator system can surpass.**

Introduction

A major advantage of electromagnetic hyperthermia is the ability to control constructive and destructive interference in locations removed from the antenna applicator. Ideally, focusing power on a

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